Intersection of a line and a circle

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Abstract

This short note details the required computation to get the intersection point(s) of a circle and a line in 2D Euclidean geometry, using two methods.

1 Introduction

A circle with a radius r and centered at (x_0, y_0) is defined by

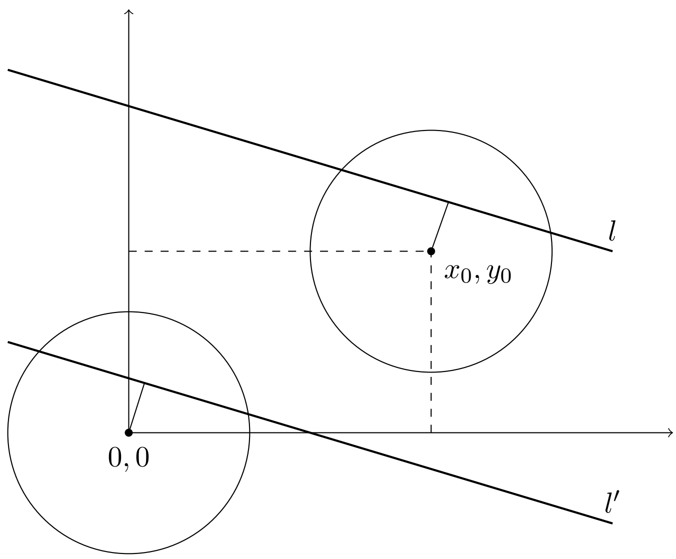
$$(x - x_0)^2 + (y - y_0)^2 - r = 0 \tag{1}$$

A line l expressed with homogeneous coordinates is defined by:

$$l : ax + by + c = 0 \tag{2}$$

2 Translating to origin

In order to simplify calculations, we translate the circle at $(0, 0)$ and compute the intersection points with the adjusted line l'. The latter is computed from l by adjusting c. Once we have the intersection points for the translated circle, we get the real points coordinates by adding to the computed solution the translation (x_0, y_0).



The adjusted line l' has the same slope as l , thus its expression can be written $a\,x + b\,y + c' = 0$, and we only need to compute the value of c'.

The distance d between line l and the center of the circle $(x0, y0)$ must be equal to the distance between origin and the line l'. The distance d is given by:

$$d = \frac{a\,x_0 + b\,y_0 + c}{\sqrt{a^2+b^2}} \tag{3}$$

The distance d' between line l' and the center of the translated circle is:

$$d' = \frac{c'}{\sqrt{a^2+b^2}} \tag{4}$$

Thus we have $c' = a\,x_0 + b\,y_0 + c$

In the next sections, we will compute the intersection between line l' and circle defined by $x^2+y^2-r=0$

3 Algebraic direct method

The line l' can be written as: $y = K_1\,x + K_2$ with $K_1 = -a/b, K_2 = -c'/b$

So we have:

$$y^2 = K_1^2\,x^2 + 2\,K_1K_2\,x + K_2^2 \tag{5}$$

We insert this expression into the circle equation:

$$x^2 + K_1^2\,x^2 + 2\,K_1K_2\,x + K_2^2 - r = 0 \tag{6}$$

We can write this as: $K_Ax^2 + K_B\,x + K_C = 0$, with:

$$\begin{cases} K_A = 1 + K_1^2 \\ K_B = 2\,K_1K_2 \\ K_C = K_2^2 - r \end{cases} \tag{7}$$

This quadratic equation is solved by computing Δ:

$$\begin{aligned} \Delta &= K_B^2 - 4K_AK_C \\ &= 4K_1^2K_2^2 - 4\cdot(1+K_1^2)(K_2^2-r) \\ &= 4K_1^2K_2^2 - 4\cdot(K_2^2 - r + K_1^2K_2^2 - K_1^2\,r) \\ &= 4(r + K_1^2\,r - K_2^2) \end{aligned} \tag{8}$$

If $\Delta < 0$, there are no intersection points. If $\Delta >= 0$, the two solutions are[1];

$$x_1 = \tfrac{-K_B-\sqrt{\Delta}}{2\,K_A} \qquad x_2 = \tfrac{-K_B+\sqrt{\Delta}}{2\,K_A} \tag{9}$$

or:

$$x_1 = \tfrac{1}{2}\cdot\tfrac{-2\,K_1K_2-\sqrt{\Delta}}{1+K_1^2} \qquad x_2 = \tfrac{1}{2}\cdot\tfrac{-2\,K_1K_2+\sqrt{\Delta}}{1+K_1^2} \tag{10}$$

Once we have computed these, we can compute the two solutions for y:

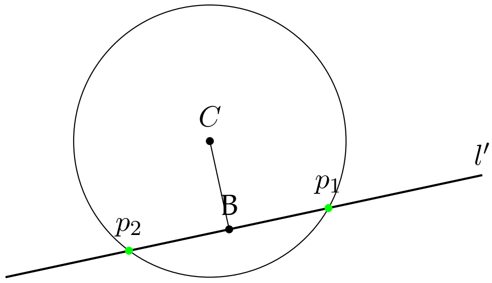
$$y_1 = K_1x_1 + K_2 \qquad y_2 = K_1x_2 + K_2 \tag{11}$$

However, this solution is practically unusable: for vertical lines, we have $b = 0$, thus the values K_1 and K_2 can not be handled correctly by a computer.

[1] Of course, if $\Delta = 0$, the two computed solutions will have an equal value, which means the intersection point is a tangent point.

4 Geometric method

The technique presented here[2] does not suffer from numeric weaknesses, thus it is the best approach. The idea is to consider the distance between the center of the circle, and B, the closest point on the line.

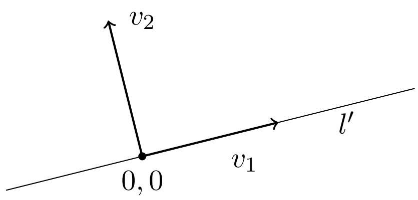


The distance $d_0 = d(BC)$ is given by:

$$d_0 = \frac{c'}{\sqrt{a^2 + b^2}} \tag{12}$$

We can already determine if there are any intersection points: if $d_0 > r$, there is no intersection. Else, we can search the coordinates of the point B. We use the line-supporting vector.

Support vector



Any line $ax+by+c = 0$ has as supporting vector $v_1 = [-b, a]$, and a perpendicular line has as supporting vector $v_2 = [a, b]$.

Thus we can state that the point B lies on line l_2, defined by $-bx + ay + c_2 = 0$, at a distance d_0 from origin. Here, the perpendicular line goes through (0,0), thus we have $c_2 = 0$. The point $B = (x_B, y_B)$ is the intersection of line l_2 and l', so we can write:

$$\begin{cases} l_2 : & -b\ x_B + a\ y_B = 0 \\ l' : & a\ x_B + b\ y_B + c' = 0 \end{cases} \tag{13}$$

Solving this brings:

$$x_B = -\frac{a\ c'}{a^2 + b^2} \qquad y_B = -\frac{b\ c'}{a^2 + b^2} \tag{14}$$

Now that we have the coordinates of B, the last step is about computing the coordinates of p_1 and p_2. These three points lie on the line l' and we have $d(B, p_1) = d(B, p_2) = d$. This distance can be computed by considering the right-angled triangle $\widehat{CBp_1}$: we have $d(C, p_1) = r$ and $d(B, p_1) = d_0$, thus $d^2 = r^2 - d_0^2$.

As we know that p_1 and p_2 lie on line l' having as support vector $[-b, a]$, we can get their coordinates by starting from point B and extending that vector with length d:

[2] Source: `https://cp-algorithms.com/geometry/circle-line-intersection.html`

For a line with a support vector $[dx, dy]$, the formulae giving the coordinate of a point $pt2 = (x_2, y_2)$ located at a distance d from a point $pt1 = (x_1, y_1)$ is:

$$\begin{cases} x_2 = x_1 + d_x \dfrac{d}{\sqrt{d_x^2 + d_y^2}} \\ y_2 = y_1 + d_y \dfrac{d}{\sqrt{d_x^2 + d_y^2}} \end{cases} \tag{15}$$

In the present case, we have two points, the second one can be computed by substracting instead of adding in the above expression. Thus, the two points p_1 and p_2 have as coordinates:

$$\begin{cases} x_1 = x_B + m\ b \\ y_1 = y_B - m\ a \end{cases} \qquad \begin{cases} x_2 = x_B - m\ b \\ y_2 = y_B + m\ a \end{cases} \qquad \text{with} \qquad m = \sqrt{\frac{d^2}{a^2 + b^2}} \tag{16}$$

Finally, we get the true coordinates of the intersection points by translating back the points, using (x_0, y_0).